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We know from (37) that the term in f'' will disappear; and it still further turns out that every alternate term after that vanishes, and we have, changing a to x,

$$f(x+h) = fx + hf'(x+\tfrac{1}{2}h) + \frac{1}{2^2}\frac{h^3}{3!}f'''(x+\tfrac{1}{2}h) + \dots$$

$$+ \frac{1}{2^{2n}}\frac{h^{2n+1}}{(2n+1)!}f^{2n+1}(x+\tfrac{1}{2}h) + \dots \tag{44}$$

This will be found useful, in the following form:

$$\frac{1}{h}\int_x^{x+h} fx\,dx = f(x+\tfrac{1}{2}h) + \tfrac{1}{3!}(\tfrac{1}{2}h)^2 f''(x+\tfrac{1}{2}h)$$

$$+ \tfrac{1}{5!}(\tfrac{1}{2}h)^4 f^{\text{iv}}(x+\tfrac{1}{2}h) + \dots \tag{45}$$

The series (44) was given by Dr. McClintock in his Essay on the Calculus of Enlargement.*

In (44) put $x=0$, then put $h=x$; whence

$$fx = f(0) + xf'(\tfrac{1}{2}x) + \frac{1}{2^2}\frac{x^3}{3!}f'''(\tfrac{1}{2}x) + \dots$$

$$+ \frac{1}{2^{2n}}\frac{h^{2n+1}}{(2n+1)!}f^{2n+1}(\tfrac{1}{2}x) + \dots, \tag{46}$$

a formula more convergent than Bernouilli's, and which we will find useful further on, in the form†

$$\frac{1}{x}\int_0^x fx\,dx = f(\tfrac{1}{2}x) + \frac{1}{3!}(\tfrac{1}{2}x)^2 f''(\tfrac{1}{2}x) + \dots$$

$$+ \frac{1}{(2n+1)!}(\tfrac{1}{2}x)^{2n} f^{2n}(\tfrac{1}{2}x) + \dots \tag{47}$$

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* See American Journal of Mathematics, Vol. II, No. 2, p. 122, where it is shown that this series may also be found by subtracting the development of $f(x+\frac{1}{2}h)$ by Taylor's formula, from that of $f(x-\frac{1}{2}h)$, and then writing $(x+\frac{1}{2}h)$ for x. It is also there shown how the series may be derived through the symbolic operation of the Calculus of Enlargement.

† This form was used by Fourier (Théorie de la Chaleur, p. 228. Paris, 1822) in his celebrated deduction of the expansion of an arbitrary function in terms of sines of the variable.

In (46) it may be shown that the general term is, if r be odd,

$$\frac{1}{2^{r-1}}\frac{x^r}{r!}\left[r + \frac{r(r-1)}{2!} + \frac{r(r-1)(r-2)}{3!} + \dots \right.$$

$$\left. + \frac{r(r-1)\dots(r-\frac{1}{2}(r+3))}{(\frac{1}{2}(r-1))!} - 2^{r-1}\right] = \frac{1}{2^{r-1}}\frac{x^r}{r!};$$

and if r be even,

$$\frac{1}{2^{r-1}}\frac{x^r}{r!}\left\{\begin{array}{l} r + \dfrac{r(r-1)(r-2)}{3!} + \dfrac{r(r-1)(r-2)(r-3)(r-4)}{5!} + \dots \\ + \dfrac{r!\,r!}{(\frac{1}{2}r)!\,(\frac{1}{2}r)!} + \dots + \dfrac{r(r-1)(r-2)}{3!} + r - 2^{r-1} \end{array}\right\} = 0.$$

5) Let $a_1 = a + p^{-1}h$; then we have

$$f(x+h) = fx + hf'(x + p^{-1}h) + \tfrac{1}{2}p^{-1}(p-2)\,h^2 f''(u), \tag{48}$$

which has an interesting geometrical illustration.

If $a_0 = 0$, $a_1 = p^{-1}x$, $a_2 = q^{-1}x$; then

$$fx = f(0) + xf'(p^{-1}x) + \frac{p-2}{p}\frac{x^2}{2!}f''(q^{-1}x)$$

$$+ \left[1 - 3\frac{p-2}{pq} - \frac{3}{p^2}\right]\frac{x^3}{3!}f'''(u). \tag{49}$$

6) Let $x = y + nh$, $a_0 = y$, $a_1 = y + h$, $a_r = y + rh$; then we have the following formula:*

$$f(y+nh) = fy + nh\,f'(y+h) + n(n-2)\frac{h^2}{2!}f''(y+2h) + \dots$$

$$+ n(n-r)^{r-1}\frac{h^r}{r!}f^r(x+rh) + \dots \tag{50}$$

And if $y = 0$, and $nh = x$,

$$f(x) = f(0) + xf'\left[\frac{x}{n}\right] + \frac{n-2}{n}\frac{x^2}{2!}f''\left[\frac{2x}{n}\right] + \dots$$

$$+ \left[\frac{n-r}{n}\right]^{r-1}\frac{x^r}{r!}f^r\left[\frac{r-1}{n}x\right] + \dots \tag{51}$$

* This may be transformed into the series of Abel, which will be deduced further on. It is usually stated of Abel's series, that it is true for functions developable in powers of e^x. See Carr's Synopsis, p. 282. This point will be brought up again in Section III.

7) Let $x = y + nh$, $a_0 = y$, $a_1 = y - h$, $a_r = y - rh$; then

$$f(y + nh) = fy + nh\,f'(y - h) + n(n + 2)\frac{h}{2!}f''(y - 2h) + \dots$$
$$+ n(n + r)^{r-1}\frac{h^r}{r!}f^r(y - rh) + \dots \tag{52}$$

Compare the deduction of this, here given, with that given in Carr's Synopsis, p. 358, where it is obtained through the aid of symbolic operation; also, note there the misprint in the last term.

8) Let $x = y + h$, $a_0 = y$, $a_1 = y + h$, $a_r = y + rh$; then

$$f(x + h) = fy + hf'(y + h) - \tfrac{1}{2}h^2f''(y + 2h) + \tfrac{2}{3}h^3f'''(u), \tag{53}$$

and $y = 0$, $h = z$, gives

$$fz = f(0) + zf'(z) - \tfrac{1}{2}z^2f''(2z) + \tfrac{2}{3}z^3f'''(u). \tag{54}$$

These are particular cases of (50).

9) Let $x = y + h$, $a_0 = y$, $a_1 = y - h$, $a_r = y - rh$; then

$$f(x + h) = fy + hf'(y - h) + \tfrac{3}{2}h^2f''(y - 2h) + \tfrac{8}{3}h^3f'''(u), \tag{55}$$

and $y = 0$ gives

$$fh = f(0) + hf'(h) + \tfrac{3}{2}h^2f''(-2h) + \tfrac{8}{3}h^3f'''(u). \tag{56}$$

These interesting formulæ, (55) and (56), were given by Dr. McClintock in his Calculus of Enlargement, where they were doubtless given for the first time, deduced there by means of the symbolic operations, and possibly they have never been reached before in any other way. They are particular cases of (52), which may itself be got from (50) by changing the signs of n and h.

10) Other interesting forms may be got by combining the preceding formulæ. We notice the following:—

Add and subtract (50) and (52); thus,

$$f(y + nh) = fy + \tfrac{1}{2}nh\,[f'(y + h) + f'(y - h)] + \dots$$
$$+ \frac{n}{2}\frac{h^r}{r!}[(n - r)^{r-1}f^r(y + rh) + (n + r)^{r-1}f^r(y - rh)] + \dots, \tag{57}$$

and

$$-\int_{y-h}^{y+h} fx\,dx = \Sigma \frac{h^{r-1}}{r!}\left[(n-r)^{r-1} f^{r-1}(y+rh) - (n+r)^{r-1} f^{r-1}(y-rh)\right]. \quad (58)$$

In (44) change the sign of h, and add and subtract the result from (44). Whence,

$$y_m = m - \sum_1^{\infty} \frac{h^{2n-1}}{2^{2n-1}(2n-1)!}\left[f^{2n-1}(x+\tfrac{1}{2}h) - f^{2n-1}(x-\tfrac{1}{2}h)\right], \quad (59)$$

wherein y_m is the mid-ordinate of an arc $y = fx$ whose extreme ordinates are $2h$ apart, and m is the mid-ordinate of the chord of the arc.

Also

$$\int_{x-h}^{x+h} fx\,dx = \sum_0^{\infty} \frac{h^{2n+1}}{2^{2n}(2n+1)!}\left[f^{2n}(x+\tfrac{1}{2}h) + f^{2n}(x-\tfrac{1}{2}h)\right]. \quad (60)$$

Compare these last two equations with the theorems of Stirling and Boole.

11) For sake of future reference, we write down as an application of (46), in which we put $f = e^x$,

$$\frac{e^x - 1}{xe^{\frac{1}{2}x}} = 1 + \frac{1}{3!}\left[\frac{x}{2}\right]^2 + \frac{1}{5!}\left[\frac{x}{2}\right]^4 + \ldots$$

12) In trying to get certain forms for a special purpose, I computed a number of partial series after the order of the Bernouilli formula, the first few terms of some of which I give here because their appearance is interesting. They all have reference to definite integration, and may be increased in number indefinitely. Possibly some very pretty series may be obtained in this way.

If, $a_0 = 0$, $a_1 = 0$, $a_2 = a_3 = \ldots = a_n = \frac{1}{3}x$, then

$$fx = f(0) + xf'(0) + \tfrac{1}{2!}x^2 f''(\tfrac{1}{3}x) + \tfrac{1}{3}\cdot\tfrac{1}{4!}x^4 f^{\text{iv}}(\tfrac{1}{3}x) + \tfrac{2}{27}\cdot\tfrac{1}{5!}x^5 f^{\text{v}}(\tfrac{1}{3}x)$$
$$+ \tfrac{1}{9}\cdot\tfrac{1}{6!}x^6 f^{\text{vi}}(u). \quad (62)$$

If, $a_0 = 0$, $a_1 = \frac{1}{2}x$, $a_2 = \frac{1}{3}x$, $a_3 = \frac{1}{4}x$, $a_4 = \frac{1}{5}x$, then

$$fx = f(0) + xf'(\tfrac{1}{2}x) + \tfrac{1}{4!}x^3 f'''(\tfrac{1}{4}x) + \tfrac{1}{4}\cdot\tfrac{1}{4!}x^4 f^{\text{iv}}(\tfrac{1}{5}x) + \tfrac{9}{32}\cdot\tfrac{1}{5!}x^5 f^{\text{v}}(u). \quad (63)$$

If, $a_0 = 0$, $a_1 = \frac{1}{2}x$, $a_3 = \frac{1}{3}x$, $a_4 = \frac{1}{4}x$, then

$$fx = f(0) + xf'(\tfrac{1}{2}x) + \tfrac{1}{4!}x^3 f'''(\tfrac{1}{3}x) + \tfrac{1}{3!\cdot 4!}x^4 f^{\text{iv}}(\tfrac{1}{4}x) + \tfrac{29}{144}\cdot\tfrac{1}{5!}x^5 f^{\text{v}}(u). \quad (64)$$

If, $a_0 = 0$, $a_1 = 0$, $a_2 = \frac{2}{3}x$, $a_3 = \frac{1}{4}x$, then

$$fx = f(0) + xf'(0) + \tfrac{1}{2}x^2f''(\tfrac{2}{3}x) + \tfrac{1}{3}x^3f'''(\tfrac{1}{4}x) - \tfrac{1}{9}x^4f^{\text{iv}}(u). \tag{65}$$

If, $a_0 = 0$, $a_1 = 0$, $a_2 = \frac{2}{3}x$, $a_3 = 5x/4\,!$, then

$$fx = f(0) + xf'(0) + \tfrac{1}{2}x^2f''(\tfrac{2}{3}x) + \tfrac{1}{3}x^3f'''(5x/4\,!) - \tfrac{3541}{432}\cdot\tfrac{1}{5!}x^5f^{\text{v}}(u). \tag{66}$$

If, $a_0 = 0$, $a_1 = 0$, $a_2 = \frac{2}{3}x$, $a_3 = \frac{1}{3}x$, then

$$fx = f(0) + xf'(0) + \tfrac{1}{2}x^2f''(\tfrac{2}{3}x) + \tfrac{1}{3}x^3f'''(\tfrac{1}{3}x) - \tfrac{14}{3}\cdot\tfrac{1}{4!}x^4f^{\text{iv}}(u). \tag{67}$$

If, $a_0 = 0$, $a_1 = \frac{1}{2}x$, $a_2 = 2^{-2}x$, $a_3 = 2^{-3}x$, $a_4 = 2^{-4}x$, then

$$fx = f(0) + xf'(\tfrac{1}{2}x) + \tfrac{1}{3}\cdot\tfrac{1}{8}x^3f'''(2^{-3}x) + \tfrac{1}{4}\cdot\tfrac{1}{16}x^4f^{\text{iv}}(2^{-4}x) + \tfrac{17}{32}\cdot\tfrac{1}{5!}x^5f^{\text{v}}(u). \tag{68}$$

13) Of particular interest are the logarithmic formulæ which flow from (22), and its special forms. The general formula comes directly from (23); thus,

$$\log x = \log a_0 - \frac{A_1}{a_1} - \frac{A_2}{a_2^2} - 2\,!\,\frac{A_3}{a_3^3} - \ldots - (n-1)\,!\,\frac{A_n}{a_n^n} - n\,!\,\frac{A_{n+1}}{u^{n+1}} \tag{69}$$

This is a formula which is exceedingly flexible. We may if desired put $a_0 = 1$ and thus eliminate $\log a_0$. The formula for $\log x$ contains in its first term one arbitrary, in its second two, and so on; so that in r terms we have at our disposal r arbitrary constants. We may thus derive a number of logarithmic formulæ by giving specific values to these arbitraries, knowing that we shall obtain convergent series under the conditions set forth in § 10, at least.

We notice only a few of the immediately apparent of these, rather more to verify the general formula than with any other object in view.

Of course, if $a_0 = a_1 = \ldots = a_n$,

$$\log(a_0^{-1}x) = a_0^{-1}(x - a_0) - \tfrac{1}{2}a_0^{-2}(x - a_0)^2 + \tfrac{1}{3}a_0^{-3}(x - a_0)^3 - \ldots, \tag{70}$$

in which $a_0 = 1$ gives

$$\log x = (x - 1) - \tfrac{1}{2}(x - 1)^2 + \tfrac{1}{3}(x - 1)^3 - \ldots, \tag{71}$$

and $x = 1$, $a_0 = y$ gives

$$\log y = (1 - y^{-1}) + \tfrac{1}{2}(1 - y^{-1})^2 + \ldots, \tag{72}$$

all well known forms.

$x = 1$ and $h = y - 1$ in (44) give at once the familiar

$$\tfrac{1}{2}\log y = h(h+2)^{-1} + \tfrac{1}{3}h^3(h+2)^{-3} + \tfrac{1}{5}h^5(h+2)^{-5} + \dots \quad (73)$$

From (50), we get

$$\log(y + nh) = \log y + nh(y+h)^{-1} - \tfrac{1}{2}n(n-2)h^2(y+2h)^{-2} + \dots, \quad (74)$$

or

$$\log(1 + nhy^{-1}) = nh(y+h)^{-1} - \tfrac{1}{2}(1-2n^{-1})n^2h^2(y+2h)^{-2}$$
$$+ \tfrac{1}{3}(1-3n^{-1})^2 n^3h^3(y+3h)^{-3} + \dots \quad (75)$$

In this put $nhy^{-1} = z$,

$$\log(1+z) = z(1+n^{-1}z)^{-1} - \tfrac{1}{2}(1-2n^{-1})z^2(1+2n^{-1}z)^{-2}$$
$$+ \tfrac{1}{3}(1-3n^{-1})z^3(1+3n^{-1}z)^{-3} - \dots, \quad (76)$$

or if $n^{-1} = a$,

$$\log(1+z) = z(1+az)^{-1} - 2^{-1}(1-2a)z^2(1+2az)^{-2}$$
$$+ 3^{-2}(1-3a)^2z^3(1+3az)^{-3} - \dots \quad (77)$$

$a = 0$ in this last gives Mercator's well known series

$$\log(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \dots$$

$n = 1$, in (76) gives

$$\log(1+z) = z(1+z)^{-1} - \tfrac{1}{2}z^2(1+2z)^{-2} + \tfrac{1}{3}2^2z^3(1+3z)^{-3}$$
$$- \tfrac{1}{4}3^3z^4(1+4z)^{-4} + \dots \quad (78)$$

A curious formula is gotten by putting $n = z$ in (76), whence*

$$\log(1+z) = \frac{z}{2} - \frac{z(z-2)}{2\,.\,3^2} + \frac{z(z-3)^2}{3\,.\,4^3} - \frac{z(z-4)^3}{4\,.\,5^4} + \dots \quad (79)$$

* In connection with these logarithmic forms see Laurent, Traité d'Analyse, Tome III, p. 386; also Abel's works.

Writing of Abel's series, Laurent says " étudiée par M. Halphen, a été reconnue inexacte dans un grandes nombres de cas; en particulier, quand $\varphi(x) = \log x$, le second membre, loin de représentes $\log(a+x)$, représente une transcendante nouvelle. Quoi qu'il en soit, M. Bertrand démontre, dans son Traité de Calcul differentiel, que, si la fonction $\varphi(z)$ est dévelopable suivant les puissances de e^z, la formule aura lieu pour cette fonction."

16. Formula (22) may be exhibited in another form, which for some purposes is to be preferred to (22). Thus

$$\begin{vmatrix} fx & , 1, & (x-a_0), & \frac{(x-a_1)^2}{2!}, & \frac{(x-a_2)^3}{3!}, & \ldots, & \frac{(x-a_n)^{n+1}}{(n+1)!} \\ fx_0 & , 1, & (x_0-a_0), & \frac{(x_0-a_1)^2}{2!}, & \frac{(x_0-a_2)^3}{3!}, & \ldots, & \frac{(x_0-a_n)^{n+1}}{(n+1)!} \\ f'x_1 & , 0, & 1, & (x_1-a_1), & \frac{(x_1-a_2)^2}{2!}, & \ldots, & \frac{(x_1-a_n)^n}{n!} \\ f''x_2 & , 0, & 0, & 1, & (x_2-a_2), & \ldots, & \frac{(x_2-a_n)^{n-1}}{(n-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n x_n & , 0, & 0, & 0, & 0, & \ldots, & 1, (x_n-a_n) \\ f^{n+1}(u), & 0, & 0, & 0, & 0, & \ldots, & 0, \quad 1 \end{vmatrix} = 0. \tag{80}$$

Let, $x_0 = a_0$, $x_1 = a_1$, $x_2 = a_2$, etc.; then

$$\begin{vmatrix} fx & , 1, & (x-a_0), & \frac{(x-a_1)^2}{2!}, & \frac{(x-a_2)^3}{3!}, & \ldots, & \frac{(x-a_n)^{n+1}}{(n+1)!} \\ fa_0 & , 1, & 0, & \frac{(a_0-a_1)^2}{2!}, & \frac{(a_0-a_2)^3}{3!}, & \ldots, & \frac{(a_0-a_n)^{n+1}}{(n+1)!} \\ f'a_1 & , 0, & 1, & 0, & \frac{(a_1-a_2)^2}{2!}, & \ldots, & \frac{(a_1-a_n)^n}{n!} \\ f''a_2 & , 0, & 0, & 1, & 0, & \ldots, & \frac{(a_2-a_n)^{n-1}}{(n-1)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n a_n & , 0, & 0, & 0, & 0, & \ldots, & 1, \quad 0 \\ f^{n+1}(u), & 0, & 0, & 0, & 0, & \ldots, & 0, \quad 1 \end{vmatrix} = 0. \tag{81}$$

Expanding this with respect to the first column, we have

$$fx = fa_0 - A_1 f'a_1 + A_2 f'a_2 - \ldots + (-1)^n A_n f^n a_n + \ldots, \tag{82}$$

in which the a's may be computed from the recurrence formula

$$A_r = \frac{(-1)^r}{r!}\left[(x-a_{r-1})^r - (a_0 - a_{r-1})^r\right] + \frac{(-1)^r}{(r-1)!}(a_1 - a_{r-1})^{r-1} A_1$$

$$+ \frac{(-1)^{r+1}}{(r-2)!}(a_2 - a_{r-1})^{r-2} A_2 + \ldots + \tfrac{1}{3!}(a_{r-3} - a_{r-1})^3 A_{r-3}$$

$$- \tfrac{1}{2!}(a_{r-2} - a_{r-1})^2 A_{r-2}. \tag{83}$$

In (82) and (83), make the substitutions

$$x - a_r = \sum_0^r b_r, \quad a_p - a_r = \sum_{p+1}^r b_r .$$

Then we get

$$fx = f(x - b_0) + C_1 f'(x - b_0 - b_1) + \ldots + C_n f^n(x - \sum_0^n b_r) + \ldots, \tag{84}$$

a formula published by Mr. Glashan as " An Extension of Taylor's Theorem," in the American Journal of Mathematics, Vol. I, No. 3.*

The coefficient C_r may be computed from (83) after making the substitutions, the first three are

$$1!\, C_1 = b_0 ,$$
$$2!\, C_2 = b_0^2 + 2b_0 b_1 ,$$
$$3!\, C_3 = b_0^3 + 3b_0^2 (b_1 + b_2) + 3b_0 (b_1^2 + 2b_1 b_2) .$$

Put $b_0 = a$, $b_1 = b_2 = \ldots = b_n = \beta$, and deduce as did Mr. Glashan, the series of Abel,

$$fx = f(x - a) + af'(x - a - \beta) + \frac{a(a + 2\beta)}{2!} f''[x - (a + 2\beta)] + \ldots . \tag{85}$$

The form in which Abel gives it, however, is†

$$f(x + a) = fx + a f'(x + \beta) + \frac{a(a - 2\beta)}{2!} f''(x + 2\beta)$$
$$+ \frac{a(a - 3\beta)^2}{3!} f'''(x + 3\beta) + \ldots . \tag{86}$$

Compare this with series (50).

17. The celebrated Legendrian coefficient,‡

$$X_n = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n ,$$

is of the nth degree; it therefore belongs to the class of rational integral functions with which we are now dealing.

* In a subsequent paper (Am. Jour. Math., Vol. IV, No. 3) Mr. Glashan remarks that this series is due to Professor Cayley, he having given a proof of it in *Solutions of the Cambridge Senate-House Problems*, 1848–51, pp. 94–96.

† See Oeuvres Complètes de Niels Abel, Tome II, p. 73, where Abel deduces this expansion by means of generating functions.

‡ Called also Laplace's coefficient of a single variable, also Kugelfunctionen, fonctions sphériques, and spherical harmonics. See Todhunter's Treatise on Laplace's, Lamé's, and Bessel's Functions.

Let $X_{n,a}^{r}$ be the rth derivative of the nth Legendrian, in which after differentiation we replace x by the arbitrary constant a. Then, we have

$$\begin{vmatrix} fx & , 1 , & X_1 & , X_2 & , \ldots , & X_{n+1} \\ fy & , 1 , & X_{1,y} & , X_{2,y} & , \ldots , & X_{n+1,y} \\ f'a & , 0 , & X'_{1,a} & , X'_{2,a} & , \ldots , & X'_{n+1,a} \\ f''b & , 0 , & 0 & , X''_{2,b} & , \ldots , & X''_{n+1,b} \\ f'''c & , 0 , & 0 & , 0 & , \ldots , & X'''_{n+1,c} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n}z & , 0 , & 0 & , 0 & , \ldots , & X^{n}_{n+1,z} \\ f^{n+1}(u), & 0 , & 0 & , 0 & , \ldots , & 1 \end{vmatrix} = 0 . \tag{87}$$

We notice particularly the case of the general formula, wherein

$$y = a = b = c = \ldots = z = 0 .$$

Use, after McClintock, the symbols $x_b^{n)}$ and $x_b^{n(}$ to denote the upper and lower factorial products respectively

$$x\,(x+b)\,(x+2b)\ldots(x+\overline{n-1}\,.\,b)\,,$$

$$x\,(x-b)\,(x-2b)\ldots(x-\overline{n-1}\,.\,b)\,.$$

Make the above substitution in (87), and consider the nth Legendrian. Differentiate it n times and put $x = 0$ in the results, beginning with the function itself. These are, in order, the constituents of the nth column. After factoring out the diagonal constituent, the nth column, from the diagonal up, will be found to be

$$1, 0, -\frac{1}{2_2^{1)}(2n-1)_2^{1(}}, 0, \frac{1}{2_2^{2)}(2n-1)_2^{2(}}, 0, -\frac{1}{2_2^{3)}(2n-1)_2^{3(}}, \ldots, \frac{2^{n-1}(n-1)!}{(2n-1)!}X_n.$$

If n be even, the next to the last term is

$$\frac{(-1)^{\frac{n}{2}}}{2_2^{\frac{n}{2})}(2n-1)_2^{\frac{n}{2}(}} .$$

If n be odd, the next to the last term is zero, and the next to the last but one is

$$\frac{(-1)^{\frac{n-1}{2}}}{2_2^{\frac{n-1}{2})}(2n-1)_2^{\frac{n-1}{2}(}} .$$

Therefore, we have

$$\begin{vmatrix} fx & , 1, & X_1, & \frac{X_2}{3}, & \frac{1}{3.5}X_3, & \frac{1}{3.5.7}X_4, & \ldots, & \frac{2^n n!\,X_{n+1}}{(2n+1)!} \\ f0 & , 1, & 0, & \frac{-1}{2.3}, & 0, & \frac{1}{2.4.5.7}, & \ldots, & \ldots \\ f'0 & , 0, & 1, & 0, & \frac{-1}{2.5}, & 0, & \ldots, & \ldots \\ f''0 & , 0, & 0, & 1, & 0, & \frac{-1}{2.7}, & \ldots, & \ldots \\ f'''0 & , 0, & 0, & 0, & 1, & 0, & \ldots, & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n0 & , 0, & 0, & 0, & 0, & 0, & \ldots, & 0 \\ f^{n+1}(u), & 0, & 0, & 0, & 0, & 0, & \ldots, & 1 \end{vmatrix} = 0. \tag{88}$$

Write out (88) with respect to the first column, getting

$$fx = f0 - C_1 f'0 + C_2 f''0 - \ldots + (-1)^n C_n f^n 0 + \ldots + R, \tag{89}$$

in which we have for the coefficient C_r

$$C_r = \begin{vmatrix} 1, & X_1, & \frac{X_2}{3}, & \frac{1}{3.5}X_3, & \ldots, & \frac{2^{r-1}(r-1)!}{(2r-1)!}X_r \\ 1, & 0, & \frac{-1}{2.3}, & 0, & \ldots, & \ldots \\ 0, & 1, & 0, & \frac{-1}{2.5}, & \ldots, & \ldots \\ 0, & 0, & 1, & 0, & \ldots, & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & 0, & \ldots,\ 1, & 0 \end{vmatrix}, \tag{90}$$

in which the last column is to be filled in, as the general column is formed above, according as r is even or odd. The above value of C_r writes out in the recurrence formula,

$$C_r = \frac{C_{r-2}}{2_2^{1}(2r-1)_2^{1}} - \ldots + \frac{(-1)^{m+1}C_{r-2m}}{2_2^{m}(2r-1)_2^{m}} + \ldots$$
$$+ (-1)^r \frac{2^{r-1}(r-1)!\,X_r}{(2r-1)!}. \tag{91}$$

It is easy to see that if r is odd, C_r contains no even Legendrians; and if even, no odd ones.

We have

$$C_0 = 1\,,$$

$$C_1 = -x\,,$$

$$C_2 = \frac{1}{3}X_2 + \frac{1}{2\,.\,3} = \frac{1}{2\,!}x^2\,,$$

$$C_3 = -\frac{1}{3\,.\,5}X_3 + \frac{1}{2\,.\,3}C_1 = -\frac{1}{3\,!}x^3\,,$$

$$C_4 = \frac{1}{3\,.\,5\,.\,7}X_4 + \frac{1}{2\,.\,7}C_2 - \frac{1}{2\,.\,4\,.\,5\,.\,7} = \frac{1}{4\,!}x^4\,,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\;;$$

and we may show by induction that $C_r = (-1)^r x^r/r\,!$; so that (89) brings us back again to Maclaurin's form. Equating the determinant (90) to $(-1)^r x^r/r\,!$, we have an expression for any integral power of x in terms of Legendrians, which when written out gives

$$(-1)^r x^r/r! = a_r X_r + a_{r-2}X_{r-2} + a_{r-4}X_{r-4} + \ldots\,,$$

where the a's are the minors of the Legendrians in the first row. Since we know (see Todhunter's Functions, p. 15) that

$$(-1)^m \frac{2^{m-1}}{m(2m-1)\,!}\,a_m = \frac{2m+1}{2}\int_{-1}^{+1} X_m\, x^r dx\,,$$

the above gives the value of this integral expressed as a determinant.

While the preceding result is interesting, it is not what we wish, as our object is to obtain the expansion of an arbitrary function in terms of Legendre's coefficients. To effect this we proceed to change the determinant (88) into a shape which we shall frequently employ hereafter. In the first place, for the sake of brevity in printing, let us temporarily put

$$r_p = \frac{1}{2_2^{\,p} r_2^{\,p}}\,.$$

Treat the determinant (88) according to the method of (13), and consider Fx. Run the first column and row to the middle, the odd columns to the right, the even ones to the left, the odd rows to the bottom, and the even rows to the top, thus obtaining for Fx the determinant

$$\begin{vmatrix} 1 & -5_1 & +9_2 & \ldots & \pm(2n-3)_{n-1} & f'0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & -9_1 & \ldots & \pm(2n-3)_{n-2} & f'''0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & \pm(2n-3)_{n-3} & f^{\text{v}}0 & 0 & 0 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & f^{2n-1}0 & 0 & 0 & 0 & \ldots & 0 \\ X_1 & \frac{1}{3.5}X_3 & \frac{1}{3.5.7.9}X_5 & \ldots & \frac{2^{2n-3}(2n-3)!}{(4n-3)!}X_{2n-1} & fx & \frac{1}{3}X_2 & \frac{1}{3.5.7}X_4 & \frac{1}{3.5.7.9.11}X_6 & \ldots & \frac{2^{2n}(2n)!}{(4n-1)!}X_{2n} \\ 0 & 0 & 0 & \ldots & 0 & f''0 & 1 & -7_1 & 11_2 & \ldots & \pm(2n-1)_{n-1} \\ 0 & 0 & 0 & \ldots & 0 & f^{\text{iv}}0 & 0 & 1 & -11_1 & \ldots & \pm(2n-1)_{n-2} \\ 0 & 0 & 0 & \ldots & 0 & f^{\text{vi}}0 & 0 & 0 & 1 & \ldots & \pm(2n-1)_{n-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & f^{2n}0 & 0 & 0 & 0 & \ldots & 1 \end{vmatrix} . \quad (92)$$

By aid of this we write

$$fx = A_0 - A_1X_1 + \frac{1}{3}A_2X_2 - \frac{1}{3.5}A_3X_3 + \ldots$$
$$+ (-1)^n \frac{2^{n-1}(n-1)!}{(2n-1)!} A_nX_n + R. \quad (93)$$

In which we have for the coefficients A_r,

$$(-1)^r A_r = \begin{vmatrix} -(2r+3)_1, & +(2r+5)_2, & -(2r+7)_3, & \ldots, & f^r0 \\ 1 & , -(2r+5)_1, & +(2r+7)_2, & \ldots, & f^{r+2}0 \\ 0 & , 1 & , -(2r+7)_1, & \ldots, & f^{r+4}0 \\ 0 & , 0 & , 1 & \ldots, & f^{r+6}0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & , 0 & , 0 & , \ldots, & 1, f^n0 \end{vmatrix} . \quad (94)$$

To determine A_0, which is the Fy of (13), we merely put $x=0$ in (93); whence

$$A_0 = f0 - \frac{1}{2.3}A_2 - \frac{1}{2_2^{2}7_2^{2}}A_4 - \frac{1}{2_2^{3}11_2^{3}}A_6 - \ldots - \frac{1}{2_2^{n}(2n-1)_2^{n}}A_{2n} - \ldots,$$

in which the A's are the same as given by (94).

We notice, in connection with the first expansion, that the minor of $f^{2n}0$ in the determinant (92) is the equivalent of $x^{2n}/(2n)!$, and the minor of $f^{2n-1}0$ is the value of $x^{2n-1}/(2n-1)!$ (See Todhunter's Functions).

18. Consider the Hypergeometric series

$$1 - \frac{\alpha_1 \beta_1}{\gamma_1} \frac{x}{1\,!} + \frac{\alpha_1^{1(}\beta_1^{)1}}{\gamma_1^{)1}} \frac{x^2}{2\,!} - \ldots + (-1)^{m+1} \frac{\alpha_1^{m(}\beta_1^{)m}}{\gamma_1^{)m}} \frac{x^{m+1}}{(m+1)\,!} + \cdots,$$

which is always convergent for $x < 1$.

For brevity put $\rho^{)m} = \beta_1^{)m}/\gamma_1^{)m}$. Give α the values 1, 2, 3, etc., in succession, thus forming the functions,

$$H_1 = 1 - \rho x\,,$$

$$H_2 = 1 - 2\rho x + 2\,!\,\rho^{)1} \frac{x^2}{2\,!},$$

$$H_3 = 1 - 3\rho x + 3^{1(}\rho^{)1} \frac{x^2}{2\,!} - 3\,!\,\rho^{)2} \frac{x^3}{3\,!},$$

. . . .

$$H_r = 1 - r\rho x + r^{1(}\rho^{)1} \frac{x^2}{2\,!} - \ldots + (-1)^{m+1} r^{m(}\rho^{)m} \frac{x^{m+1}}{(m+1)\,!} + \ldots + r\,!\,\rho^{)r-1} \frac{x^r}{r\,!}.$$

Substitute the functions $H_r/r\,!$ in the composite, and put all of the arbitraries equal to zero, and remove the common factors from the rows; whence

$$\begin{vmatrix} fx & 1 & H_1 & \frac{H_2}{2\,!} & \frac{H_3}{3\,!} & \frac{H_4}{4\,!} & \ldots & \frac{H_{n+1}}{(n+1)\,!} \\ f0 & 1 & 1 & \frac{1}{2\,!} & \frac{1}{3\,!} & \frac{1}{4\,!} & \ldots & \frac{1}{(n+1)\,!} \\ -\frac{f'0}{\rho} & 0 & 1 & 1 & \frac{1}{2\,!} & \frac{1}{3\,!} & \ldots & \frac{1}{n\,!} \\ +\frac{f''0}{\rho^{)1}} & 0 & 0 & 1 & 1 & \frac{1}{2\,!} & \ldots & \frac{1}{(n-1)\,!} \\ -\frac{f'''0}{\rho^{)2}} & 0 & 0 & 0 & 1 & 1 & \ldots & \frac{1}{(n-2)\,!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{(-1)^n}{\rho^{)n-1}} f^n 0 & 0 & 0 & 0 & 0 & 0 & \ldots\,,\ 1 & 1 \\ f^{n+1}(u) & 0 & 0 & 0 & 0 & 0 & \ldots\,,\ 0 & 1 \end{vmatrix} = 0\,. \qquad (95)$$

Therefore,

$$fx = A_0 - A_1 H_1 + \frac{1}{2\,!} A_2 H_2 - \frac{1}{3\,!} A_3 H_3 + \ldots + (-1)^n \frac{1}{n\,!} A_n H_n + R, \quad (96)$$

wherein

$$(-1)^r A_r = \begin{vmatrix} +\dfrac{f^r0}{\rho^{r-1}} & ,1, & \dfrac{1}{2!}, & \dfrac{1}{3!}, & \dfrac{1}{4!}, & \ldots, & \dfrac{1}{n!} \\ -\dfrac{f^{r+1}0}{\rho^{r}} & ,1, & 1\ , & \dfrac{1}{2!}, & \dfrac{1}{3!}, & \ldots, & \dfrac{1}{(n-1)!} \\ +\dfrac{f^{r+2}0}{\rho^{r+1}} & ,0, & 1\ , & 1\ , & \dfrac{1}{2!}, & \ldots, & \dfrac{1}{(n-2)!} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \dfrac{(-1)^n}{\rho^{r+n-1}} f^{r+n}0, & 0\ , & 0\ , & 0\ , & 0\ , & \ldots, & 1 \end{vmatrix}$$

$$= \frac{1}{\rho^{r-1}} f^r0 + \frac{1}{\rho^r} f^{r+1}0 + \frac{1}{2!\,\rho^{r+1}} f^{r+2}0 + \ldots + \frac{1}{n!\,\rho^{r+n-1}} f^{r+n}0,$$

in virtue of (36). To get A_0, we make $x = 0$ in (96), noticing that all of the H's then become unity; whence

$$A_0 = f0 + A_1 - \frac{1}{2!} A_2 + \ldots + (-1)^{n+1} \frac{1}{n!} A_n + \ldots$$

18. Consider Bessel's Function

$$J_n(x) = \frac{x^n}{2^n n!} \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\,.\,4(2n+2)(2n+4)} - \ldots \right],$$

$$= \sum_{r=0}^{r=\infty} \frac{(-1)^r}{r!\,(n+r)!} \left[\frac{x}{2} \right]^{n+2r},$$

a series always convergent.

We have

$$\begin{vmatrix} fx & , J_0(x)\ , & J_1(x)\ , & J_2(x)\ , & J_3(x)\ , & J_4(x)\ , & J_5(x)\ , & \ldots, & J_{n+1}(x) \\ f0 & ,\ 1\ , & 0\ , & 0\ , & 0\ , & 0\ , & 0\ , & \ldots, & 0 \\ 2^1 f'0 & ,\ 0\ , & 1\ , & 0\ , & 0\ , & 0\ , & 0\ , & \ldots, & 0 \\ 2^2 f''0 & , \dfrac{-2!}{1!\,1!}, & 0\ , & 1\ , & 0\ , & 0\ , & 0\ , & \ldots, & 0 \\ 2^3 f'''0 & ,\ 0\ , & \dfrac{-3!}{1!\,2!}, & 0\ , & 1\ , & 0\ , & 0\ , & \ldots, & 0 \\ 2^4 f^{\mathrm{iv}}0 & , \dfrac{+4!}{2!\,2!}, & 0\ , & \dfrac{-4!}{1!\,3!}, & 0\ , & 1\ , & 0\ , & \ldots, & 0 \\ 2^5 f^{\mathrm{v}}0 & ,\ 0\ , & \dfrac{+5!}{2!\,3!}, & 0\ , & \dfrac{-5!}{1!\,4!}, & 0\ , & 1\ , & \ldots, & 0 \\ 2^6 f^{\mathrm{vi}}0 & , \dfrac{-6!}{3!\,3!}, & 0\ , & \dfrac{+6!}{2!\,4!}, & 0\ , & \dfrac{-6!}{1!\,5!}, & 0\ , & \ldots, & 0 \\ 2^7 f^{\mathrm{vii}}0 & ,\ 0\ , & \dfrac{-7!}{3!\,4!}, & 0\ , & \dfrac{+7!}{2!\,5!}, & 0\ , & \dfrac{-7!}{1!\,6!}, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n+1}(u), & 0\ , & 0\ , & 0\ , & 0\ , & 0\ , & 0\ , & \ldots, & 1 \end{vmatrix} = 0. \quad (97)$$

The $(n+2)$th row being

$$2^n f^n 0,\, 0,\, \frac{\pm\, n!}{(\tfrac{1}{2}n-\tfrac{1}{2})!\,(\tfrac{1}{2}n+\tfrac{1}{2})!},\, 0, \ldots,\, 0, -C_{n,3},\, 0, +C_{n,2},\, 0, -C_{n,1},\, 0, 1, 0, 0 \ldots,$$

or

$$2^n f^n 0,\, 0,\, \frac{\pm\, n!}{(\tfrac{1}{2}n)!\,(\tfrac{1}{2}n)!},\, 0, \ldots,\, 0, -C_{n,3},\, 0, +C_{n,2},\, 0, -C_{n,1},\, 0, 1, 0, 0, \ldots,$$

according as n is odd or even respectively. $C_{n,r}$ meaning the number of combinations of n things taken r at a time, or simply the binomial coefficients.

Expand (97) according to form (13), and arrange Fx thus

$$\left|\begin{array}{cccccccccccc}
1, & 0, & 0, & 0, & 0, & \ldots, & 2^1 f' 0, & 0, & 0, & 0, & 0, & 0, \ldots \\
-C_{3,1}, & 1, & 0, & 0, & 0, & \ldots, & 2^3 f''' 0, & 0, & 0, & 0, & 0, & 0, \ldots \\
+C_{5,2}, & -C_{5,1}, & 1, & 0, & 0, & \ldots, & 2^5 f^{v} 0, & 0, & 0, & 0, & 0, & 0, \ldots \\
-C_{7,3}, & +C_{7,2}, & -C_{7,1}, & 1, & 0, & \ldots, & 2^7 f^{vii} 0, & 0, & 0, & 0, & 0, & 0, \ldots \\
+C_{9,4}, & -C_{9,3}, & +C_{9,2}, & -C_{9,1}, & 1, & \ldots, & 2^9 f^{ix} 0, & 0, & 0, & 0, & 0, & 0, \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
J_1(x), & J_3(x), & J_5(x), & J_7(x), & J_9(x), & \ldots, & fx, & J_0(x), & J_2(x), & J_4(x), & J_6(x), & J_8(x), \ldots \\
0, & 0, & 0, & 0, & 0, & \ldots, & f0, & 1, & 0, & 0, & 0, & 0, \ldots \\
0, & 0, & 0, & 0, & 0, & \ldots, & 2^2 f'' 0, & -C_{2,1}, & 1, & 0, & 0, & 0, \ldots \\
0, & 0, & 0, & 0, & 0, & \ldots, & 2^4 f^{iv} 0, & +C_{4,2}, & -C_{4,1}, & 1, & 0, & 0, \ldots \\
0, & 0, & 0, & 0, & 0, & \ldots, & 2^6 f^{vi} 0, & -C_{6,3}, & +C_{6,2}, & -C_{6,1}, & 1, & 0, \ldots \\
0, & 0, & 0, & 0, & 0, & \ldots, & 2^8 f^{viii} 0, & +C_{8,4}, & -C_{8,3}, & +C_{8,2}, & -C_{8,1}, & 1, \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right| . \qquad (98)$$

Whence we obtain, expanding with respect to the middle row,

$$fx = A_0 J_0(x) - A_1 J_1(x) + A_2 J_2(x) - \ldots + (-1)^n A_n J_n(x) + R, \qquad (99)$$

in which we have

$$A_r = \left|\begin{array}{ccccc}
f0, & 1, & 0, & \ldots, & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
2^r f^r 0, & \cdot & \cdot & \ldots, & \dfrac{-r!}{1!\,(r-1)!}
\end{array}\right|$$

$$= -C_{r,1} A_{r-2} - C_{r,2} A_{r-4} - \ldots - C_{r,m} A_{r-2m} - \ldots + (-1)^r 2^r f^r 0.$$

The series may be verified to any extent by arranging according to powers of x, and thus identifying by means of Maclaurin's form. We know (see Todhunter's Functions) that any integral power of x may be expressed in terms of Besselian functions; we may therefore expect to find $x^r/r!$ equivalent to the coefficient of f^r0, in the expansion of (98) with respect to its middle column, which is expressed in terms of an infinite series of Besselian functions written in the form of a determinant. Expanding (98) with respect to the middle column, we get

$$fx = A_0 f0 - 2'A_1 f'0 + 2^2 A_2 f''0 + \ldots + (-1)^r 2^r A_r f^r 0 + R, \quad (100)$$

wherein

$$(-1)^r A_r = \begin{vmatrix} J_r(x), & -C_{r+2,1}, & +C_{r+4,2}, & -C_{r+6,3}, & \ldots \\ J_{r+2}(x), & 1, & -C_{r+4,1}, & +C_{r+6,2}, & \ldots \\ J_{r+4}(x), & 0, & 1, & -C_{r+6,1}, & \ldots \\ J_{r+6}(x), & 0, & 0, & 1, & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

Further discussion of the expansion of fx in terms of Bessel's functions is reserved for later consideration, when a different method of treatment will be applied.

19. This section, while treating rational integral functions as the type of those functions which yield a body-determinant unity, includes also certain transcendental functions whose coefficients may be put in recurrence formulæ. Examples of these will be given below.

20. To expand fx in terms of the central-factorial functions

$$\begin{aligned} x^{(2r+1)} &= (x - ar)\ldots(x - a)\,x\,(x + a)\ldots(x + ra), \\ &= x(x^2 - a^2)(x^2 - 2^2a^2)\ldots(x^2 - r^2a^2), \\ &= x^{2r+1} - a^2P_{r,1}\,x^{2r-1} + a^4P_{r,2}\,x^{2r-3} - \ldots + (-1)^{r+1}a^{2r-2}P_{r,r-1}x^3 \\ &\quad + (-1)^r a^{2r}(r!)^2 x, \end{aligned}$$

wherein $P_{r,p}$ is used to represent the sum of the products, without repetition, of the quantities $1, 2^2, \ldots, r^2$ taken p at a time, and a is an arbitrary constant.

Substitute in the composite the functions

$$\frac{1}{(2r+1)!}x^{(2r+1)},$$

and put $x = 0$ after differentiation. Remembering in virtue of theorem § 4 that only the odd derivative rows will appear, we have

$$\begin{vmatrix} fx & , 1 , & x , & \frac{1}{3!}x^{)3(} , & \frac{1}{5!}x^{)5(} , & \frac{1}{7!}x^{)7(} & , \ldots , & \ldots , & \ldots \\ f0 & , 1 , & 0 , & 0 \; , & 0 \; , & 0 & , \ldots , & \ldots , & \ldots \\ f'0 & , 0 , & 1 , & -\frac{1}{6}a^2 , & +\frac{1}{30}a^4 , & -\frac{1}{140}a^6 , & \ldots , & \ldots , & \ldots \\ f'''0 & , 0 , & 0 , & 1 \; , & -\frac{1}{4}a^2 \; , & +\frac{7}{120}a^4 , & \ldots , & \ldots , & \ldots \\ f^{\mathrm{v}}0 & , 0 , & 0 , & 0 \; , & 1 \; , & -\frac{1}{3}a^2 & , \ldots , & \ldots , & \ldots \\ f^{\mathrm{vii}}0 & , 0 , & 0 , & 0 \; , & 0 \; , & 1 & , \ldots , & \ldots , & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{2n-1}0 , & 0 , & 0 , & 0 \; , & 0 \; , & 0 & , \ldots , & 1 \; , & \ldots \\ \Phi(u) & , 0 , & 0 , & 0 \; , & 0 \; , & 0 & , \ldots , & 0 \; , & 1 \end{vmatrix} = 0 , \quad (101)$$

wherein the column headed with $x^{)2r+1(}$ is, from the diagonal up,

$$1, -\frac{(2r-1)!}{(2r+1)!}a^2 P_{r,1}, +\frac{(2r-3)!}{(2r+1)!}a^4 P_{r,2}, \ldots, \frac{(-1)^{r+1}3!}{(2r+1)!}a^{2r-2}P_{r,r-1},$$
$$\frac{(-1)^r a^{2r}(r!)^2}{(2r+1)!}, 0, \frac{x^{)2r+1(}}{(2r+1)!}.$$

If we expand with respect to the first column, we express the coefficient of $f^{(2p-1)}0$ in finite terms of the factorial functions. If we expand with respect to the factorial functions, we derive the formula

$$fx = f0 + A_1 x + A_3\frac{x^{)3(}}{3!} + \ldots + A_{2n+1}\frac{x^{)2n+1(}}{(2n+1)!} + \ldots, \qquad (102)$$

wherein the coefficient A_{2r+1} is an infinite series of the form

$$A_{2r+1} = C_1 f^{2r+1}(0) + C_2 f^{2r+3}(0) + \ldots.$$

The coefficients C being numbers which may be computed directly from the determinant, and may be connected by a recurrence formula.

The function fx should, in general, be an odd function.

21. In like manner we may consider the expansion of fx in terms of the upper factorial functions

$$x^{(r)} = x(x+a)(x+2a)\ldots(x+\overline{r-1}a),$$
$$= x^r + P_{r,1}ax^{r-1} + P_{r,2}a^2x^{r-2} + \ldots + P_{r,r-1}a^{r-1}x^2 + a^r(r-1)!\,x,$$

(which becomes the lower factorial $x^{)r(}$ if a is negative) wherein now $P_{r,p}$ is the sum of the products, without repetition, of the quantities $1, 2, \ldots, r-1$ taken p at a time.

Substituting in the composite the functions

$$\frac{1}{r!}x^{)r},$$

since

$$\left[\frac{d^p x^{)r}}{dx^p}\right]_{x=0} = a^{r-p}p!\,P_{r,r-p}\,,$$

we have

$$\begin{vmatrix} fx & , 1, & x, & \frac{1}{2!}x^{)2}, & \frac{1}{3!}x^{)3}, & \frac{1}{4!}x^{)4} & , \ldots, \ldots \\ f0 & , 1, & 0, & 0\ , & 0\ , & 0 & , \ldots, \ldots \\ f'0 & , 0, & 1, & \frac{1}{2}a\ , & \frac{1}{3}a^2, & \frac{1}{4}a^3 & , \ldots, \ldots \\ f''0 & , 0, & 0, & 1\ , & a\ , & \frac{11}{12}a^2 & , \ldots, \ldots \\ f'''0 & , 0, & 0, & 0\ , & 1\ , & \frac{3}{2}a & , \ldots, \ldots \\ f^{\text{iv}}0 & , 0, & 0, & 0\ , & 0\ , & 1 & , \ldots, \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Phi(u) & , 0, & 0, & 0\ , & 0\ , & 0 & , \ldots, 1 \end{vmatrix} = 0\,. \qquad (103)$$

The column headed with $x^{)r}$ being

$$1\,,\ \frac{r-1}{2}a\,,\ \frac{(r-2)!}{r!}a^2P_{r,2}\,,\ \ldots,\ \frac{2!}{r!}a^{r-1}P_{r,r-1}\,,\ \frac{a^r}{r!}\,.$$

We have from (103)

$$fx = f0 + \sum_1^n A_r\frac{x^{)r}}{r!}\,, \qquad (104)$$

the same remarks applying to the coefficient A_r as in the preceding article.

If we put $a = 0$, we have, of course, Maclaurin's formula. These interesting factorial series will be taken up again later on.

22. We close the section with a few examples of transcendental functions which may be given as illustrations of the application of the method of section II to this class of functions.

We know that

$$(\sin^{-1}ax)^n = (ax)^n + A_2\,(ax)^{n+2} + A_4\,(ax)^{n+4} + \ldots,$$

where the coefficients A are numbers to be obtained by equating the coefficients of c in the two series

$$e^{c\sin^{-1}ax} = 1 + cax + \frac{(cax)^2}{2!} + \frac{c(c^2+1)}{3!}(ax)^3 + \frac{c^2(c^2+2^2)}{4!}(ax)^4 + \ldots,$$

$$e^{c\sin^{-1}ax} = ax + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots.$$

Making the substitutions, and putting $x = 0$ after differentiation, we have, letting $\theta_x^n = (\sin^{-1}ax)^n$,

$$\begin{vmatrix} fx \;, & 1\,, & \theta_x \;, & \theta_x\,, & \theta_x \;, & \theta_x^4\,, & \ldots, & \theta_x^{n+1} \\ fy \;, & 1\,, & \theta_y \;, & \theta_y^2\,, & \theta_y \;, & \theta_y\,, & \ldots, & \theta_y^{n+1} \\ f'0\,, & 0\,, & 1 \;, & 0\,, & 0 \;, & 0\,, & \ldots, & 0 \\ \dfrac{f''0}{2!}\,, & 0\,, & 0 \;, & 1\,, & 0 \;, & 0\,, & \ldots, & 0 \\ \dfrac{f'''0}{3!}\,, & 0\,, & \dfrac{1}{2.3} \;, & 0\,, & 1 \;, & 0\,, & \ldots, & 0 \\ \dfrac{f^{\text{iv}}0}{4!}\,, & 0\,, & 0 \;, & \dfrac{2^2}{3.4}\,, & 0 \;, & 1\,, & \ldots, & 0 \\ \dfrac{f^{\text{v}}0}{5!}\,, & 0\,, & \dfrac{1.3}{2.4.5}, & 0\,, & \dfrac{3^2+1}{4.5}\,, & 0\,, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Phi(u), & 0\,, & 0 \;, & 0\,, & 0 \;, & 0\,, & \ldots, & 1 \end{vmatrix} = 0. \qquad (105)$$

Therefore

$$fx - fy = \sum_{r=1}^{r=n} A_r[(\sin^{-1}ax)^r - (\sin^{-1}ay)^r]$$

wherein we have for the first five values of A,

$$A_1 = f'0\,,$$
$$A_2 = \tfrac{1}{2!}f''0\,,$$
$$A_3 = -\tfrac{1}{6}f'0 + \tfrac{1}{3!}f'''0\,,$$
$$A_4 = +\tfrac{1}{6}f''0 - \tfrac{1}{4!}f^{\text{iv}}0\,,$$
$$A_5 = +\tfrac{1}{10}f'0 - \tfrac{1}{3!}f'''0 + \tfrac{1}{5!}f^{\text{v}}0\,.$$

and so on, the general value being expressed as a recurrence formula.

23. Consider the functions

$$\sin(r\sin^{-1}ax) \text{ and } \cos(r\sin^{-1}ax)$$

which for brevity we write $\sin r\theta$ and $\cos r\theta$ respectively.

Represent by the symbol $r^{)n(}$, the central factorial product

$$(r-n)\ldots(r-1)\,r\,(r+1)\ldots(r+n)\,,$$

and by $r_0^{)2n-1(}$ and $r_e^{)2n(}$ the products, respectively,

$$(r-2n+1)\ldots(r-3)(r-1)\,r\,(r+1)(r+3)\ldots(r+2n-1)\,,$$

$$(r-2n)\ldots(r-4)(r-2)\,r\,(r+2)(r+4)\ldots(r+2n)\,.$$

Make the substitutions as below indicated in the composite, putting $x=0$ after differentiation, computing the derivatives from the formulæ

$$\sin(r\sin^{-1}ax)=\Sigma\, r_0^{)2n-1(}\,\frac{x^{2n-1}}{(2n-1)!}\,,$$

$$\cos(r\sin^{-1}ax)=\Sigma\, r_e^{)2n(}\,\frac{x^{2n}}{(2n)!}\,,$$

and assuming

$$r^{)-1(}=r \text{ and } r^{)0(}=1\,.$$

We write down then for Fx the determinant

$$\left|\begin{array}{lcccccccc}
fx & ,\ \sin\theta\,, & \dfrac{\cos 2\theta}{2} & ,\ \sin 3\theta, & \dfrac{\cos 4\theta}{4} & ,\ldots, & \sin(2n-1)\,\theta & , & \dfrac{\cos 2n\theta}{2n} \\
\dfrac{1}{a}f'0 & ,\ 1\ , & 0 & ,\ 3\ , & 0 & ,\ldots, & (2n-1) & , & 0 \\
\dfrac{-f''0}{2a^2} & ,\ 0\ , & 1 & ,\ 0\ , & 2 & ,\ldots, & 0 & , & \dfrac{(2n)}{2} \\
\dfrac{+f'''0}{3_0^{)1(}a^3} & ,\ 0\ , & 0 & ,\ 1\ , & 0 & ,\ldots, & \dfrac{(2n-1)_0^{)1(}}{3_0^{)1(}} & , & 0 \\
\dfrac{-f^{\mathrm{iv}}0}{4_e^{)2(}a^4} & ,\ 0\ , & 0 & ,\ 0\ , & 1 & ,\ldots, & 0 & , & \dfrac{(2n)_e^{)2(}}{4_e^{)2(}} \\
\dfrac{+f^{\mathrm{v}}0}{5_0^{)3(}a^5} & ,\ 0\ , & 0 & ,\ 0\ , & 0 & ,\ldots, & \dfrac{(2n-1)_0^{)3(}}{5_0^{)3(}} & , & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\dfrac{\pm f^{2n-1}0}{(2n-1)_0^{)2n-3(}\,a^{2n-1}} & ,\ 0\ , & 0 & ,\ 0\ , & 0 & ,\ldots, & \dfrac{(2n-1)_0^{)2n-3(}}{(2n-1)_0^{)2n-3(}} & , & 0 \\
\dfrac{\mp f^{2n}0}{(2n)_e^{)2n-2(}a^{2n}} & ,\ 0\ , & 0 & ,\ 0\ , & 0 & ,\ldots, & 0 & , & \dfrac{(2n)^{)2n-2(}}{(2n)^{)2n-2(}}
\end{array}\right|$$

Whence we derive the formula

$$fx-fy=\sum_{r=1}^{r=n}\Big\{A_{2r-1}[\sin(\overline{2r-1}\sin^{-1}ax)-\sin(\overline{2r-1}\sin^{-1}ay)]$$
$$-\frac{1}{2r}A_{2r}[\cos(2r\sin^{-1}ax)-\cos(2r\sin^{-1}ay)]\Big\}\,; \qquad (106)$$

wherein

$$A_{2r-1} = \begin{vmatrix} \dfrac{f^{2n-1}(0)}{(2n-1)_0^{)2n-3(}a^{2n-1}}, & \dfrac{(2n+1)_0^{)2n-3(}}{(2n-1)_0^{)2n-3(}}, & \dfrac{(2n+3)_0^{)2n-3(}}{(2n-1)_0^{)2n-3(}}, & \dfrac{(2n+5)_0^{)2n-3(}}{(2n-1)_0^{)2n-3(}}, & \cdots \\ \dfrac{f^{2n+1}(0)}{(2n+1)_0^{)2n-1(}a^{2n+1}}, & 1 \quad , & \dfrac{(2n+3)_0^{)2n-1(}}{(2n+1)_0^{)2n-1(}}, & \dfrac{(2n+5)_0^{)2n-1(}}{(2n+1)_0^{)2n-1(}}, & \cdots \\ \dfrac{f^{2n+3}}{(2n+3)^{)2n+1(}a^{2n+3}}, & 0 \quad , & 1 \quad , & \dfrac{(2n+5)_0^{)2n+1(}}{(2n+3)_0^{)2n+1(}}, & \cdots \\ \dfrac{f^{2n+5}(0)}{(2n+5)^{)2n+3(}a^{2n+5}}, & 0 \quad , & 0 \quad , & 1 \quad , & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

and

$$A_{2r} = -\begin{vmatrix} \dfrac{f^{2n}(0)}{(2n)_e^{)2n-2(}a^{2n}}, & \dfrac{(2n+2)_e^{)2n-2(}}{(2n)_e^{)2n-2(}}, & \dfrac{(2n+4)_e^{)2n-2(}}{(2n)_e^{)2n-2(}}, & \dfrac{(2n+6)_e^{)2n-2(}}{(2n)_e^{)2n-2(}}, & \cdots \\ \dfrac{f^{2n+2}(0)}{(2n+2)_e^{2n}a^{2n+2}}, & 1 \quad , & \dfrac{(2n+4)_e^{)2n(}}{(2n+2)^{)2n(}}, & \dfrac{(2n+6)_e^{)2n(}}{(2n+2)_e^{)2n(}}, & \cdots \\ \dfrac{f^{2n+4}(0)}{(2n+4)_e^{2n+2}a^{2n+4}}, & 0 \quad , & 1 \quad , & \dfrac{(2n+6)_e^{)2n+2(}}{(2n+4)_e^{)2n+2(}}, & \cdots \\ \dfrac{f^{2n+6}(0)}{(2n+6)_e^{2n+4}a^{2n+6}}, & 0 \quad , & 0 \quad , & 1 \quad , & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

simpler forms of (167), which may be obtained through specifying the arbitraries are obvious.

24. Consider the expansion of fx in terms of the functions

$$\varphi_r(x) = x^r e^{ax} \cos bx\,,$$

whose nth derivative is, when $x = 0$ after differentiation,

$$\varphi_r{}^n(0) = n_{-1}^{r(} (a^2 + b^2)^{\frac{n-r}{2}} \cos\left[(n-r)\tan^1(b/a)\right]$$

or zero, according as n is greater than or less than r.

For brevity, put $a^2 + b^2 = c^2$, and $\theta = \tan^{-1}(b/a)$. Using $n_{-1}^{r(}$, or more simply $n^{r(}$, since the difference -1 is always the same, to denote the descending factorial product

$$n\,(n-1)\,(n-2)\ldots(n-r+1)\,.$$

Making the substitutions in the composite and reducing to simplest form by factoring, we have

$$\left|\begin{array}{llllllll}
fx & ,\ 1\ , & \varphi_1(x) & ,\ \varphi_2(x) & ,\ \varphi_3(x) & ,\ \ldots, & \varphi_n(x)\ , & \varphi_{n+1}(x) \\
fy & ,\ 1\ , & \varphi_1(y) & ,\ \varphi_2(y) & ,\ \varphi_3(y) & ,\ \ldots, & \varphi_n(y), & \varphi_{n+1}(y) \\
f'(0) & ,\ 0\ , & 1 & ,\ 0 & ,\ 0 & ,\ \ldots & 0\ , & 0 \\
\frac{f''(0)}{2!} & ,\ 0\ , & \frac{c\cos}{1!} & ,\ 1 & ,\ 0 & ,\ \ldots, & 0\ , & 0 \\
\frac{f'''(0)}{3!} & ,\ 0\ , & \frac{c^2\cos 2\theta}{2!} & ,\ \frac{c\cos\theta}{1!} & ,\ 1 & ,\ \ldots, & 0\ , & 0 \\
\frac{f^{\mathrm{iv}}(0)}{4!} & ,\ 0\ , & \frac{c^3\cos 3\theta}{3!} & ,\ \frac{c^2\cos 2\theta}{2!} & ,\ \frac{c\cos\theta}{1!} & ,\ \ldots, & 0\ , & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{f^{n}(0)}{n!} & ,\ 0\ , & \frac{c^{n-1}\cos(n-1)\theta}{(n-1)!} & ,\ \frac{c^{n-2}\cos(n-2)\theta}{(n-2)!} & ,\ \frac{c^{n-3}\cos(n-3)\theta}{(n-3)!} & ,\ \ldots, & 1\ , & 0 \\
\Phi(u) & ,\ 0\ , & 0 & ,\ 0 & ,\ 0 & ,\ \ldots, & 0\ , & 1
\end{array}\right| = 0. \quad (107)$$

Whence

$$fx - fy = \sum_{r=1}^{r=n} A_r\,(x^r e^{ax}\cos bx - y^r e^{ay}\cos by)\,,$$

wherein

$$A_r = (-1)^{r+1}\left|\begin{array}{lllll}
f'(0) & ,\ 1 & ,\ 0 & ,\ \ldots, & 0 \\
\frac{f''(0)}{2!} & ,\ c\cos\theta & ,\ 1 & ,\ \ldots, & 0 \\
\frac{f'''(0)}{3!} & ,\ \frac{c^2\cos 2\theta}{2!} & ,\ c\cos\theta & ,\ \ldots, & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{f^{r}(0)}{r!} & ,\ \frac{c\ \ \cos(r-1)\theta}{(r-1)!} & ,\ \frac{c^{-}\cos(r-2)\theta}{(r-2)!} & ,\ \ldots, & c\cos\theta
\end{array}\right|.$$

Writing this out, we have

$$A_r = -c\cos\theta\,.\,A_{r-1} + \frac{1}{2!}c^2\cos 2\theta\,.\,A_{r-2} - \ldots$$

$$+ (-1)^{r-1}\frac{1}{(r-1)!}c^{r-1}\cos(r-1)\theta\,.\,f'(0) + \frac{1}{r!}f^{r}(0)\,,$$

and $A_1 = f'(0)$, from which the other values follow at once.

25. Expand the function fx according to the functions

$$\frac{1}{r!}(\varphi x)^r,$$

where we have the existing relation

$$\varphi x = \frac{x - z}{\psi x},$$

in which z is independent of φx.

Make the substitutions in the composite, and after differentiation put $x = z$, which causes all terms in the body determinant above the diagonal to vanish. The diagonal term in column headed $(\varphi x)^r/(r!)$ will be $(\varphi' z)^r$. Divide each row by this diagonal term, so as to make the diagonal unity throughout. This gives for the term below the diagonal term the value $\frac{1}{2} r (r+1) \varphi'' z/(\varphi' z)^2$, and so on, descending the column, the terms becoming more complicated as we descend; and while they are easily computed, I have not been able to determine a general law for writing them down. Use the symbol $D_z^p \varphi^r$ to mean that the function $(\varphi x)^r/(r!)$ has been differentiated p times with respect to x and in the result x replaced by z. We have

$$\begin{vmatrix} fx, & 1, & \varphi x, & \frac{(\varphi x)^2}{2!}, & \frac{(\varphi x)^3}{3!}, & \frac{(\varphi x)^4}{4!}, & \ldots, & \frac{(\varphi x)^n}{n!}, & \frac{(\varphi x)^{n+1}}{(n+1)!} \\ fz, & 1, & 0, & 0, & 0, & 0, & \ldots, & 0, & 0 \\ \frac{f'z}{\varphi' z}, & 0, & 1, & 0, & 0, & 0, & \ldots, & 0, & 0 \\ \frac{f''z}{(\varphi' z)^2}, & 0, & \frac{\varphi'' z}{(\varphi' z)^2}, & 1, & 0, & 0, & \ldots, & 0, & 0 \\ \frac{f'''z}{(\varphi' z)^3}, & 0, & \frac{\varphi''' z}{(\varphi' z)^3}, & 3\frac{\varphi'' z}{(\varphi' z)^2}, & 1, & 0, & \ldots, & 0, & 0 \\ \frac{f^{\mathrm{iv}}z}{(\varphi' z)^4}, & 0, & \frac{\varphi^{\mathrm{iv}} z}{(\varphi' z)^4}, & D_z^{\mathrm{iv}}\varphi^2, & 6\frac{\varphi'' z}{(\varphi' z)^2}, & 1, & \ldots, & 0, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{f^n z}{(\varphi' z)^n}, & 0, & \frac{D_z^n \varphi}{(\varphi' z)^n}, & \frac{D_z^n \varphi^2}{(\varphi' z)^n}, & \frac{D_z^n \varphi^3}{(\varphi' z)^n}, & \frac{D_z^n \varphi^4}{(\varphi' z)^n}, & \ldots, & 1, & 0 \\ \Phi(u), & 0, & 0, & 0, & 0, & 0, & \ldots, & 0, & 1 \end{vmatrix} = 0. \quad (108)$$

Expanding this with respect to the first row, we have

$$fx = fz + \sum_1^n A_r \frac{(\varphi x)^r}{r!}, \quad (109)$$

wherein, since

$$\varphi' z = \frac{1}{\psi z}, \quad \varphi'' z = -2\frac{\psi' z}{(\psi z)^2}, \quad \varphi''' z = \frac{6(\psi' z)^2 - 3\psi z\, \psi'' z}{(\psi z)^3},$$

we have

$$A_1 = \frac{f'z}{\varphi' z} = \psi z \, . \, f'z\,, \qquad A_2 = \frac{f'z\,\varphi''z - f''z\,\varphi'z}{(\varphi'z)^3} = -\frac{d}{dz}\left[(\psi z)^2 f'z\right],$$

$$A_3 = 3\frac{f'z\,(\varphi''z)^2}{(\varphi'z)^5} + \frac{f'''z}{(\varphi'z)^3} - \frac{f'z\,\varphi'''z}{(\varphi'z)^4} - 3\,\frac{f''z\,\varphi''z}{(\varphi'z)^4} = \frac{d^2}{dz^2}\left[(\psi z)^3 f'z\right].$$

So that we have

$$fx = fz + \varphi x\,[\psi z\, f'z] + \frac{(\varphi x)^2}{2\,!}\frac{d}{dz}\left[(\psi z)^2 f'z\right] + \frac{(\varphi x)^3}{3\,!}\frac{d^2}{dz^2}\left[(\psi z)^3 f'z\right] + \ldots,$$

which is Lagrange's theorem. I have not identified the coefficients of (109) with those of Lagrange's theorem beyond those here given. The theorem (108) seems to be the same form of Lagrange's theorem as that given by Wronski* in 1812, which has been studied by Professor Cayley, in a paper "On Wronski's Theorem," in the Quarterly Journal of Pure and Applied Mathematics, Vol. XII, No. 47, p. 221, 1873, where Professor Cayley takes up the series in a form which seems to be the same as here given, and proceeds to identify the determinant coefficients with those of Lagrange's Theorem.

To illustrate the method of this last expansion, let us effect the development of $fx = \log x^h$ in terms of the functions

$$\frac{1}{r\,!}y^r = \frac{1}{r\,!}\left(x^{h(1-a)} - x^{-ah}\right)^r.$$

Make the substitutions in the composite and after differentiation put $x = 1$, we have

$$\left|\begin{array}{cccccc}
\log x^h, & y, & \frac{y^2}{2\,!}, & \frac{y^3}{3\,!}, & \ldots, \frac{y^n}{n\,!}, & \frac{y^{n+1}}{(n+1)\,!} \\
1, & 1, & 0, & 0, & \ldots, 0, & 0 \\
-1, & -2a, & 1, & 0, & \ldots, 0, & 0 \\
+2\,!, & +3a(a+1), & -6a, & 1, & \ldots, 0, & 0 \\
-3\,!, & -4a(a+1)(a+2), & 12a(2a+1), & -12a, & \ldots, 0, & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
(-1)^n n\,!, & D_1^n(y), & D_1^n\frac{y^2}{2\,!}, & D_1^n\frac{y^3}{3\,!}, & \ldots, 1, & 0 \\
\Phi(u), & 0, & 0, & 0, & \ldots, 0, & 1
\end{array}\right| = 0\,, \quad (110)$$

* At the time this paper was written I was wholly unaware of the work which Wronski has done in the direction of the general expansion of series. Following up Professor Cayley's paper I have made it a point to study Wronski's work on series and find it so interesting that I shall devote a future portion of my paper to an examination and discussion of his work and methods of developing a function in series. See also Emile West's Exposé of Wronski's methods (Gauthier-Villars, Paris, 1886), where this theorem is treated at length.

which, when expanded, gives Dr. McClintock's very general logarithmic series (Am. Jour. Math. Vol. II, No. 2, p. 108).

$$\log x^h = y + (2a-1)^{1(}\frac{y^2}{2\,!} + (3a-1)^{2(}\frac{y^3}{3\,!} + (4a-1)^{3(}\frac{y^4}{4\,!} + \ldots + R.$$

I have deduced this form for the coefficients as far as here written only, and have not attempted to identify the coefficient of y^r in the expansion of (110) with

$$\frac{1}{r\,!}(ra-1)^{r-1(},$$

which I imagine would prove to be a matter of considerable difficulty.

26. We close these illustrations with the determination of the form which an expansion of fx in terms of integral powers of sines or cosines of the variable must have if the development be possible.

Thus, substituting in the composite the functions

$$\frac{1}{r\,!}\sin^r x\,,$$

and using $D_0{}^p s^r$ to indicate the pth derivative of $\frac{1}{r\,!}\sin^r x$ in which after differentiation x is put equal to zero, we have

$$\left|\begin{array}{llllllll} fx, & 1, & \sin x, & \frac{\sin^2 x}{2\,!}, & \frac{\sin^3 x}{3\,!}, & \frac{\sin^4 x}{4\,!}, & \ldots, & \frac{\sin^{n+1} x}{(n+1)\,!} \\ f0, & 1, & 0, & 0, & 0, & 0, & \ldots, & 0 \\ f'0, & 0, & 1, & 0, & 0, & 0, & \ldots, & 0 \\ f''0, & 0, & 0, & 1, & 0, & 0, & \ldots, & 0 \\ f'''0, & 0, & -1, & 0, & 1, & 0, & \ldots, & 0 \\ f^{\text{iv}}0, & 0, & 0, & -4, & 0, & 1, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f^n 0, & 0, & D_0{}^n s, & D_0{}^n s^2, & D_0{}^n s^3, & D_0{}^n s^4, & \ldots, & 0 \\ \Phi(u), & 0, & 0, & 0, & 0, & 0, & \ldots, & 1 \end{array}\right| = 0. \qquad (111)$$

The expansion of this gives

$$fx = f0 + \sum_1^n A_r \frac{1}{r\,!}\sin^r x + R, \qquad (112)$$

wherein the first five values of A_r are

$$A_1 = 1\,,\quad A_2 = 1,\qquad A_3 = f'0 + f'''0\,,$$
$$A_4 = 4f'' + f^{\text{iv}}0\,,\qquad A_5 = 9f'0 + 10f'''0 + f^{\text{v}}0\,,$$

and any number of them are easily computed.

Instead of $\sin^r x$ we may generalize by writing $(\sin 2\pi x/l)^r$, and after differentiation put $x = l$; or we may substitute the functions $(\cos \pi x/l)^r$, and after differentiation put $x = \frac{1}{2}\,l$.

We may also effect the expansion in terms of the functions

$$\frac{1}{(2r-1)!}\sin^{2r-1}x,\quad \frac{1}{(2r)!}\sin^{2r}x\,,\quad \frac{1}{(2r-1)!}\cos^{2r-1}x,\quad \frac{1}{(2r)!}\cos^{2r}x\,,$$

remembering that the first of these is an odd function, while the others are even. Therefore the composite of the first will contain no even derivative rows, and that of the others no odd derivative rows, in virtue of § 4.

For example, using the odd powered sines, we get

$$fx = f0 + \sin x\,f'0 + \frac{\sin^3 x}{3!}(f'0 + f'''0) + \frac{\sin^5 x}{5!}(9f'0 + 10f'''0 + f^{\text{v}}0) + \ldots\,;$$

and applying this, for sake of illustration, to expand $\sin ax$, we have

$$\sin ax = a\sin x - \frac{a(a^2-1)}{3!}\sin^3 x + \frac{a(a^2-1)(a^2-3^2)}{5!}\sin^5 x + \ldots. \quad (113)$$

If we use the even powered sines we get

$$fx = f0 + \frac{\sin^2 x}{2!}f''0 + \frac{\sin^4 x}{4!}(4f''0 + f^{\text{iv}}0) + \ldots.$$

Similarly, the expansion in cosines follows easily. These trigonometrical expansions seem of small value when compared with those of the sines and cosines of multiple angles which occur in the next section.

27. We repeat, finally, that all of the expressions in the form of series herein given are to be considered as having a finite number of terms and a terminal term R as set forth in § 3. They are therefore true equations and functions of n, the number of terms involved. These formulæ are not to be imagined as extending to infinity until it has been demonstrated that R becomes evanescent when n is infinite and the member on the right converges to fx on the left as a limit when n becomes infinitely large, and for what values of the variable the result may be relied upon as arithmetically intelligible and true. That is to say we have definitely determined the true analytical forms, whose

arithmetical values remain yet to be tested. We may thus regard the *qualitative* analysis as having been effected for these formulæ regarded as series; their *quantitative* analysis remains yet to be performed. The terminal term R is a function of an absolutely unknowable value of x, which can only be eliminated by showing that R vanishes when n is infinite. The examples given seem sufficient to illustrate the class of functions to which the section is applied, and we leave its forms for the present in order to pass now to the consideration of those more important functions which yield a difference-product as a body-determinant.

TO BE CONTINUED.

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SOLUTIONS OF EXERCISES.

325

DISCUSS the case of two forces acting together upon a particle, one force being directed to the centre, and the other directed to one focus of an ellipse, under the influence of which forces the particle freely describes the ellipse.

[*Yale Prize Problem.*]

SOLUTION.

1. Prof. Curtis, in the Messenger of Mathematics, 1880, proved that, if a particle describe a curve freely under the combined action of the forces F, F', etc., acting along r, r', etc. toward fixed centres, the equation

$$\gamma\varphi d\left[\frac{F}{\varphi}\right]+\gamma'\varphi' d\left[\frac{F'}{\varphi'}\right]+\ldots=0\,, \quad \text{or} \quad \Sigma\gamma\varphi d\left[\frac{F}{\varphi}\right]=0$$

must be satisfied at every point of the curve, φ, φ', etc. denoting the forces respectively co-directional with F, F', etc., under which *singly* the given curve would be described; and γ, γ', etc. denoting the corresponding semichords of the circle of curvature at the point.

The proof is as follows: Putting the origin at one of the centres of force, the others being (h, k), (l, m), etc., we have

$$\frac{d^2x}{dt^2}=-F\frac{x}{r}-F'\frac{x-h}{r'}-\ldots, \quad \frac{d^2y}{dt^2}=-F\frac{y}{r}-F'\frac{y-k}{r'}-\ldots.$$

Multiplying the first equation by $2\,dx$, the second by $2\,dy$, and integrating,

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=v^2=-2\,(\int Fdr+c)-2\,(\int F'dr'+c')-\ldots$$
$$=-2\,\Sigma\,(\int Fdr+c).$$

Also, since the centrifugal acceleration, v^2/ρ, at any point in the orbit, must be equal and opposite to the sum of the components of the central accelerations taken in the normal direction (ρ being the radius of curvature, and p the perpendicular from the centre of force on the tangent at any point),

$$\frac{v^2}{\rho}=F\frac{p}{r}+F'\frac{p'}{r'}+\ldots,\qquad v^2=F\frac{p}{r}\rho+F'\frac{p'}{r'}\rho+\ldots=F\gamma+F'\gamma'+\ldots.$$

(Williamson's Diff. Calc., Art. 235).

We have, therefore,

$$v^2=\Sigma F\gamma=-2\Sigma\,(\int Fdr+c).$$

Differentiating, $$\Sigma\,(Fd\gamma+\gamma dF)=-2\Sigma\,Fdr\,,$$

or $$\Sigma\{F(d\gamma+2\,dr)+\gamma\,dF\}=0\,. \qquad (1)$$

In particular, $$\varphi\,(d\gamma+2dr)+\gamma d\varphi=0\,,$$

$$d\gamma+2dr=-\frac{\gamma}{\varphi}\,d\varphi\,.$$

Substituting in (1),

$$\Sigma\gamma\left[dF-\frac{Fd\varphi}{\varphi}\right]=0\,,\quad\text{or}\quad\Sigma\gamma\varphi d\left[\frac{F}{\varphi}\right]=0\,. \qquad (2)$$

2. We will derive another equation *free from* γ.

$$\Sigma F\gamma=-2\Sigma\,(\int Fdr+c)\,. \qquad (3)$$

In particular $$\varphi\gamma=-2(\int\varphi dr+s).$$

Substituting for γ in (3),

$$\Sigma\frac{F}{\varphi}\,(\int\varphi dr+s)=\Sigma\,(\int Fdr+c)\,.$$

Differentiating,

$$\Sigma d\left[\frac{F}{\varphi}\right](\int\varphi dr+s)+\Sigma\frac{F}{\varphi}\,\varphi dr=\Sigma\,Fdr\,,\qquad\Sigma d\left[\frac{F}{\varphi}\right](\int\varphi dr+s)=0\,. \qquad (4)$$

This equation might be derived directly from (2); for we have

$$\gamma_1\varphi_1 = v_1^2 = -2\left(\int \varphi_1 dr_1 + s_1\right).$$

Substituting for $\gamma\varphi$ in (2) we obtain (4).

The constant s in (4) must be determined by the particular circumstances of the problem.

Equations (2) and (4) are evidently satisfied if F is any multiple of φ. We have, therefore, the particular solution

$$F = l\varphi\,, \qquad F' = m\varphi'\,, \qquad F'' = n\varphi''\,, \qquad \ldots.$$

That is, if a particle will freely describe a given curve under the separate action of any one of several forces, it will freely describe the same curve when the forces act together; or when they act together, the intensity of each being multiplied by any arbitrary number, provided it is given a proper initial velocity.

3. For the case of a particle describing an ellipse under the action of two forces, one directed to a focus and the other to the centre, (4) becomes

$$\frac{d}{dr}\left[\frac{F}{\varphi}\right] dr\left(\int \varphi dr + s\right) + \frac{d}{dr'}\left[\frac{F'}{\varphi'}\right] dr'\left(\int \varphi' dr' + s'\right) = 0\,, \tag{5}$$

where $\varphi = \mu r^{-2} =$ force toward left-hand focus,
$\varphi' = \mu' r' =$ " " centre.

To determine s and s' we suppose the particle to be acted on separately by φ and φ' and to be at the right hand extremity of the major axis. Evidently $\gamma = \gamma' = \rho = \dfrac{b^2}{a}$.

In the first case, if v_0 be the velocity,

$$v_0^2 = \varphi\gamma = -2\int \varphi dr - 2s = \frac{2\mu}{r} - 2s\,,$$

$$\frac{\mu}{(a+c)^2}\frac{b^2}{a} = \frac{2\mu}{a+c} - 2s\,, \quad 2s = \frac{\mu}{a}.$$

In the second case, if v_1 be the velocity,

$$v_1^2 = \varphi'\gamma' = -2\int \varphi' dr' - 2s' = -\mu' r'^2 - 2s'\,,$$

$$\mu' a\frac{b^2}{a} = -\mu' a^2 - 2s'\,,$$

$$2s' = -\mu'(a^2 + b^2)\,.$$

To obtain a relation between r and r', we have, putting the origin at the centre,

$$r^2 = x^2 + 2cx + c^2 + y^2\,,$$

$$r'^2 = x^2 + y^2\,,$$

$$r^2 - r'^2 = 2cx + c^2\,,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

$$c^2 = a^2 - b^2\,.$$

From these equations we obtain

$$r'^2 - (r - a)^2 = b^2\,, \qquad (6)$$

$$r'dr' = (r - a)\,dr\,.$$

By substituting in (5) their values for φ, φ', s, s' and dr' we obtain

$$\frac{1}{r^2(a - r)}\frac{d}{dr}(Fr^2) = \frac{a}{r'}\frac{d}{dr'}\left[\frac{F'}{r'}\right]. \qquad (7)$$

This is satisfied by the equations

$$\frac{1}{r^2(a - r)}\frac{d}{dr}(Fr^2) = f_1\left[(r - a)^2\right] + f_2\left[(r - a)^2 + b^2\right],$$

$$\frac{a}{r'}\frac{d}{dr'}\left[\frac{F'}{r'}\right] = f_1(r'^2 - b^2) + f_2(r'^2),$$

where f_1, f_2 are any functions. If particular forms are given to these functions, corresponding forms of F and F' are obtained by clearing the equations of fractions, and integrating.

4. We will now solve for F and F' using (2).

To find a relation between γ and γ', putting the origin at the left focus, we have (Williamson's Diff. Calc. Art. 172)

$$\frac{(x - c)^2}{a^2} + \frac{y^2}{b^2} = 1\,, \quad x^2 + y^2 = r^2\,,$$

$$p^2\left[\frac{(x - c)^2}{a^4} + \frac{y^2}{b^4}\right] = \left[\frac{x(x - c)}{a^2} + \frac{y^2}{b^2}\right]^2;$$

whence

$$p^2 = \frac{b^2 r}{2a - r}\,,$$

$$\gamma = p\frac{dr}{dp} = \frac{r(2a - r)}{a}\,.$$

Now, putting the origin at the centre, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,,\quad x^2+y^2=r^2\,,\quad \frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{p^2}\,;$$

whence $$a^2+b^2-r'^2=\frac{a^2b^2}{p'^2}\,,$$

$$\gamma'=p'\frac{dr'}{dp'}=\frac{a^2+b^2-r'^2}{r'}=\frac{r\,(2a-r)}{r'}\,,\ \text{by (6)},$$

$$\frac{\gamma'}{\gamma}=\frac{a}{r'}\,.$$

We have now

$$\gamma\varphi\,\frac{d}{dr}\left[\frac{F}{\varphi}\right]dr+\gamma'\varphi'\,\frac{d}{dr'}\left[\frac{F'}{\varphi'}\right]dr'=0$$

$$\gamma'=\frac{a\gamma}{r'}\,,\quad \varphi=\frac{\mu}{r^2}\,,\quad \varphi'=\mu r'\,,\quad r'dr'=(r-a)dr\,.$$

Making the proper substitutions we obtain

$$\frac{1}{r^2(a-r)}\,\frac{d}{dr}\,(Fr^2)=\frac{a}{r'}\,\frac{d}{dr'}\left[\frac{F'}{r'}\right],$$

the same as (7). [*Joseph Bowden, Jr.*]

EXERCISES.

348

Prove that, if $0<a<\beta$,

$$\int_a^\beta \log\frac{\beta-x}{x-a}\,\frac{dx}{x}=\frac{1}{2}\left[\log\frac{\beta}{a}\right]^2.$$

[*Frank Morley.*]

349

Integrate the differential equation

$$dy=\text{arc sin }(x^2)\,dx.$$

[*Artemas Martin.*]

350

LET p_1, p_2, p_3 be the points of contact of parallel tangents to a cardioid. Let them be in positive order. Let $p_2 p_3 q_1$, $p_3 p_1 q_2$, $p_1 p_2 q_3$ be positive equilateral triangles. Prove that q_1, q_2, q_3 lie on a line parallel to the tangents.

[*Frank Morley.*]

351

THE outer coatings of two condensers, A and B, are put to earth, and their inner coatings are connected together through a galvanometer, the resistance of which is g. The capacities of the condensers are C and c, respectively. Both are charged initially to the same potential difference, V_0, and then have charges of Q_0 and q_0, respectively. Show that if the inner coatings of A and B are put to earth simultaneously through wires of no self-induction, but of resistance R and r, respectively, the charge on A after t seconds will be

$$Q = \frac{Q_0}{2\varkappa}\varepsilon^{\frac{-t(\mu+\mu')}{2m}}\left\{\left[\varkappa+\mu+\mu'-\frac{2m}{CR}\right]\varepsilon^{\frac{\varkappa t}{2m}}+\left[\varkappa-\mu-\mu'+\frac{2m}{CR}\right]\varepsilon^{\frac{-\varkappa t}{2m}}\right\},$$

where $\lambda = rCR$, $\lambda' = rcR$,

$$\mu = cr(g+R),\quad \mu' = CR(g+r),$$

$$m = CcgRr,\ \text{and}\ \varkappa^2 = 4\lambda\lambda' + (\mu-\mu')^2.$$

Show also that the whole quantity of electricity which passes through the galvanometer during the discharge, will be

$$M = \frac{Q_0\{C^2R^2(\mu\mu'-\lambda\lambda') + m^2 - mCR(\mu+\mu')\}}{C^2R^2(\mu\mu'-\lambda\lambda')} = \frac{Q_0(CR-cr)}{C(g+r+R)}.$$

It is to be noticed that $\mu\mu' - \lambda\lambda'$ can never be zero. If $CR = cr$, $M = 0$, as is the case in De Sauty's method of comparing the capacities of two condensers. In applying the expressions written above to numerical problems, one sometimes needs to know that one $\left\{\begin{array}{c}\text{microfarad}\\ \text{ohm}\\ \text{microcoulomb}\end{array}\right.$ is equivalent to $\left\{\begin{array}{l}10^{-15}\\ 10^{9}\\ 10^{-7}\end{array}\right.$ absolute electromagnetic units, and to $\left\{\begin{array}{l}9 \times 10^{5}\\ 9^{-1}\times 10^{-11}\\ 3 \times 10^{3}\end{array}\right.$ absolute electrostatic units of $\left\{\begin{array}{l}\text{capacity.}\\ \text{resistance.}\\ \text{quantity.}\end{array}\right.$

[*B. O. Peirce.*]

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